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A non-linear instability theory for a wave system in plane Poiseuille flow

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The initial-value problem for linearized perturbations is discussed, and the asymptotic solution for large time is given. For values of the Reynolds number slightly greater than the critical value, above which perturbations may grow, the asymptotic solution is used as a guide in the choice of appropriate length and time scales for slow variations in the amplitude A of a non-linear two-dimensional perturbation wave. It is found that suitable time and space variables are ϵt and $\epsilon^{\frac{1}{2}}(x+a_{1r}t)$, where t is the time, x the distance in the direction of flow, ϵ the growth rate of linearized theory and $(-a_{1r})$ the group velocity. By the method of multiple scales, A is found to satisfy a non-linear parabolic differential equation, a generalization of the time-dependent equation of earlier work. Initial conditions are given by the asymptotic solution of linearized theory.

1. Introduction

We consider an incompressible viscous fluid in steady motion at a Reynolds number R close to the critical value R_c, above which small velocity perturbations, or vorticity waves, may be amplified. If their amplitude becomes too large, a non-linear theory is required in order to follow the evolution of such perturbations and, ten years ago, Stuart (1960) suggested that the development could be treated by means of an expansion in powers of $(R-R_c)$ or of some parameter close to that Reynolds number difference. After some analysis it was deduced, amongst other results, that the time-dependent amplitude (A_1) of the leading-Fourier mode of the expansion satisfied the non-linear ordinary differential equation.

$$\frac{d}{dt}|A_1|^2 = k_1|A_1|^2+k_2|A_1|^4, \tag{1.1}$$

an equation whose validity for such systems was originally conjectured by Landau (1944). From this equation a number of inferences were drawn about the behaviour of nearly monochromatic waves in parallel shear flows.

Specifically Stuart's paper was concerned with plane Poiseuille flows in an infinite channel, and subsequently the ideas were extended and assessed mathematically by Watson (1960, 1962), Eckhaus (1965), Pekeris & Shkoller

(1967, 1969), Reynolds & Potter (1967*a*), Ellingsen, Gjevik & Palm (1970), Davey & Nguyen (1971) and others. Moreover, much related work has been done on the rotating-cylinder and Bénard mechanisms of instability; but relatively brief reference will be made to that work here, since our main concern is with non-linear vorticity waves propagating on parallel, or nearly parallel, shear flows.

It is known from the experiments of Davies & White (1928) and others that in plane Poiseuille flow turbulence of some kind can exist at Reynolds numbers (based on half the channel width and on the maximum velocity) as low as 1000 or thereabouts, whereas the linearized theory of instability (Thomas 1953) gives a value of R_c of about 5780. Consequently the linearized theory gives results radically different from those of relevant experiments. Meksyn & Stuart (1951) suggested that, even for $R < R_c$, non-linear effects might provide a threshold amplitude above which velocity perturbations could grow and stimulate turbulence; their approximate theory, which yielded an amplitude-dependent critical Reynolds number as low as 3000, was at least partly successful in providing support for this idea. The later work, which pivots around equation (1.1), provides a means of making an assessment of Meksyn & Stuart's suggestion in a more rational way, since the approximations leading to (1.1) are really under better control. The value of k_1 is known from linearized theory; Pekeris & Shkoller (1967) and Reynolds & Potter (1967*a*) provided the vital calculations of the coefficient k_2 (> 0) of (1.1). Their results showed conclusively that, at least for small values of $R_c - R$, there does exist a threshold-amplitude instability phenomenon for $R < R_c$. This principle may be regarded as established, though far more work is required to give details of the amplitude-depended critical Reynolds number. In summary, therefore, there is a class of problems, typified by plane Poiseuille flow, for which linearized theory provides a sufficient condition for instability, in that the flow is unstable for $R > R_c$; however, the condition is not necessary, since non-linear effects may provide instability for $R < R_c$.

Related, but even more difficult, problems exist in Hagen–Poiseuille flow and in plane Couette flow, since in both cases linearized theory gives no critical Reynolds number, though turbulence can certainly occur. First, but not wholly successful, attempts on these two problems have been made by Ellingsen *et al.* (1970) and by Davey & Nguyen (1971) respectively. (The difficulty in interpretation of these two papers lies in the fact that, with no value for R_c, no parameter seems to exist, for small values of which the amplitude equation can be given reasonable justification.)

As mentioned earlier the Stuart–Landau theory has also been applied to the calculation of the amplitude of Taylor vortices between rotating cylinders, and of the associated required torque. In this case the coefficients are of such signs that (1.1) yields a growth from small amplitudes to an equilibrium state. The first study (Davey 1962) was structural and non-rigorous, as was the later work of Reynolds & Potter (1967*b*) to higher order in amplitude. A rigorous theory was provided by Velte (1966), while Kirchgässner & Sorger (1969) described a related method of determining the structure of the Taylor-vortex flows. These papers were successful in providing estimates of the torque in good agreement with experiment for a range of speeds above the critical value. However, as Coles (1965), Krueger, Gross & Di Prima (1966) and Di Prima & Grannick (1969) have pointed out, there exist difficulties in applying such theories to the case of the cylinders rotating in opposite directions.

The aim of the present paper is to examine afresh the problem of plane Poiseuille flow, in order to make a better evaluation of the details of the evolution of waves in that case. In particular we wish to know the behaviour of wave systems which develop in both space and time, in contrast to the work of Stuart (1960) and Watson (1960), which is concerned with the evolution of time or spatially periodic waves, and in contrast to the work of Watson (1962), which is concerned with the evolution in space of time-periodic waves. Thus our object is to obtain a generalization of (1.1). In this sense our aim is similar to that of Di Prima, Eckhaus & Segel (1971), though the methods used are quite different. In fact two methods will be used here, both different from that of Di Prima *et al.* (1971), and they bring us to an equation which we believe to be an appropriate generalization of (1.1). The arguments used are especially relevant to plane Poiseuille flows and other 'parallel' flows. They are inappropriate to, or at any rate far less relevant for, the rotating-cylinder problem mentioned earlier.

We can see reasons for this by looking at the flow from two points of view. First, we note that experiments on Taylor vortices are usually carried out on fluid confined between two concentric cylinders of radii r_1 and r_2 of finite length l and bounded at the top and bottom by planes or a free surface. In certain circumstances, involving high Reynolds numbers, substantially different values of r_1 and r_2, substantially different angular velocities of the cylinders and/or sufficiently small values of $l/|r_1-r_2|$, it is possible that the Ekman boundary layers induced near these plane surfaces can exert an important effect on the steady flow of the confined fluid. Indeed this may be a partial cause for the difficulties mentioned earlier in applying the non-linear theory to flows between counter-rotating cylinders. It is reasonable, however, to expect that these boundary layers near the plane ends have only a small effect on the main flow properties when Davey's theory gives an adequate description of Taylor vortices, namely between the first (Taylor) and second (wavy-vortex) stability boundaries, at moderate values of the Reynolds number, and when the cylinders are long ($l \gg |r_1-r_2|$) with angular velocities in the same sense. Then we can apply the theory for infinite cylinders, subject only to the requirement that the axial wavelength $2\pi/\alpha$ be chosen to make the axial velocity zero at the end planes, and therefore to be an integral fraction of l. By allowing the amplitude $|A_1|$ of the disturbances to vary slowly with axial distance we are, in effect, allowing α to be a slowly-varying function of $|A_1|$, which contradicts the requirement that $\alpha l/2\pi$ be an integer. Experiments (e.g. Coles 1965) reveal typically 20–30 Taylor cells between the plane walls, and their number does not appear to change easily near the first stability boundary. The paper by Coles (1965) shows one example of a change occurring in the number of cells, but it is through an asymmetric disturbance; more over, flow conditions are near to the second stability boundary, which separates the Taylor-vortex régime from that of wavy vortices.

Secondly, the spatial re-cycling of the flow around the cylinders in the Couette problem means that azimuthal evolution of perturbations may largely be ignored since the flow builds up to a steady state. In the plane Poiseuille problem, on the other hand, there is no re-cycling, and one is concerned with evolution over a long spatial range. These remarks, it is hoped, set the scene for our analysis.

2. Linear theory for wave systems

Let us consider Poiseuille flow under pressure between two parallel planes, which are set a distance $2h$ apart. In laminar undisturbed flow, a uniform pressure gradient produces a velocity distribution which is independent of the streamwise co-ordinate and has its maximum value U_0 at the centre of the channel. In the following analysis, the reference length is h, the reference velocity U_0 and the reference time h/U_0. We let x denote the co-ordinate parallel to the planes and z the co-ordinate normal to them. The corresponding velocity components are u and w while ψ is the stream function and t is the time.

The governing differential equation of the two-dimensional motion is

$$\frac{\partial \zeta}{\partial t}+\frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial z}=\frac{1}{R}\nabla^2\zeta, \tag{2.1}$$

where

$$\zeta=-\nabla^2\psi, \tag{2.2}$$

$R = U_0 h/\nu$ is the Reynolds number and ν denotes the kinematic viscosity. In undisturbed laminar flow, the motion is parallel to the planes and is given by

$$\overline{u}_l \equiv \partial\psi_l/\partial z = 1-z^2. \tag{2.3}$$

A bar above a symbol denotes a mean with respect to x, while the suffix l is used to denote undisturbed laminar flow.

In the papers of Stuart and Watson the disturbances to (2.3) which they discuss have the form of travelling waves, which are periodic in x with wavelength $2\pi/\alpha$ independent both of x and of the disturbance amplitude. However the criticism can be levelled at this approach that it is too limited in restricting attention to monochromatic waves since it does not allow for a more natural development of a wave system out of some initial state. For example, let us consider the pair of waves

$$\epsilon\cos\alpha x+\epsilon\cos\alpha(1+\delta)\,x, \tag{2.4}$$

which add to give

$$2\epsilon\cos\alpha(1+\tfrac{1}{2}\delta)\,x\cos\tfrac{1}{2}\alpha\delta x. \tag{2.5}$$

If δ is small then (2.5) can be considered as a periodic wave of wavelength $2\pi/\alpha(1+\frac{1}{2}\delta)$ whose amplitude $2\epsilon\cos\frac{1}{2}\alpha\delta x$ varies slowly with x. Moreover, if $\cos\alpha(1+\frac{1}{2}\delta)\,x$ is expanded for small δ, the wavelength of $\cos\alpha x$ is incorrect, and the wave evolves 'out of phase'. We wish to develop a formulation of the non-linear instability problem which allows for these properties.

As an essential preliminary to such an analysis, however, let us consider how the linearized solution evolves when R is greater than its 'critical' value R_c, which is 5780 according to linearized theory. Suppose at $t = 0$ we make a small initial disturbance which is confined within a finite distance of the origin. So long as the disturbance remains small, the perturbation stream function $\overline{\Psi}_1$ satisfies the linear equation

$$\left(\frac{\partial}{\partial t}+(1-z^2)\frac{\partial}{\partial x}\right)\nabla^2\overline{\Psi}_1+2\frac{\partial\overline{\Psi}_1}{\partial x} = R^{-1}\nabla^4\overline{\Psi}_1, \tag{2.6}$$

with boundary conditions that $\overline{\Psi}_1$, $\partial\overline{\Psi}_1/\partial z$ vanish at $z = \pm 1$, and that $\overline{\Psi}_1 \to 0$ as $|x| \to \infty$ for all finite t; as an initial condition $\overline{\Psi}_1$ is given to be an infinitely differentiable function of x, z at $t = 0$. A formal solution can be obtained by means of the Fourier–Laplace transform

$$\hat{\Psi}_1(z;\alpha,s,R) = \int_0^\infty e^{-st}\,dt\int_{-\infty}^\infty e^{-i\alpha x}\,\overline{\Psi}_1(t,x,z;R)\,dx, \tag{2.7}$$

$$\overline{\Psi}_1 = \frac{1}{4\pi^2 i}\int_{-\infty}^\infty e^{\alpha i x}\,d\alpha\int_{\gamma-i\infty}^{\gamma+i\infty}\hat{\Psi}_1\,e^{st}\,ds, \tag{2.8}$$

where the line $s_r = \gamma$ lies to the right of any singularities of $\hat{\Psi}_1$.

A full study of (2.6) is not intended, and is indeed a formidable task; instead we shall proceed intuitively (and possibly incompletely). For any fixed α the form of the outer integrand of (2.8) depends upon the branch points and upon the poles of the inner integrand $\hat{\Psi}_1$. Branch points would occur if initially $\overline{\Psi}_1$ were not completely smooth, and may occur because of some intrinsic property of (2.6) as yet unknown. We shall, for the present, exclude them from consideration. At a pole, (2.6) has an eigensolution which may be found by solving

$$-i\alpha\left\{\left(1-z^2-\frac{is}{\alpha}\right)(\hat{\Psi}_1''-\alpha^2\hat{\Psi}_1)+2\hat{\Psi}_1\right\}+R^{-1}\{\hat{\Psi}_1^{\text{iv}}-2\alpha^2\hat{\Psi}_1''+\alpha^4\overline{\Psi}_1\} = 0, \tag{2.9}$$

subject to $\hat{\Psi}_1(\pm 1) = \hat{\Psi}_1'(\pm 1) = 0$, primes denoting differentiation with respect to z. This is the Orr–Sommerfeld eigenvalue problem, and we require non-trivial solutions with eigenvalues $s(\alpha)$ for a given R.

There is a critical Reynolds number, R_c, such that for all $R < R_c$, the real part s_r of each eigenvalue is less than zero whatever the (real) value of α; consequently, in this range of Reynolds number, all the poles lie to the left of the imaginary axis of s so that, as $t \to \infty$, their contribution to the outer integrand of (2.8) vanishes. However, for any given value of α sufficiently close to α_c, which is the critical wave-number for neutral stability at $R = R_c$ and is 1·02 approximately, one pole is known to move to the right of the imaginary axis as $R - R_c$ becomes positive. For $R > R_c$ the maximum value of the real part of s occurs at, say, $\alpha = \alpha_m$, where $\alpha_m \to \alpha_c$ as $R \to R_c$. For some $R > R_c$, and for α close to α_m we may write

$$s(\alpha) = -i\alpha_m c_m + i\bar{a}_{1r}\beta - \bar{a}_2\beta^2 + \bar{a}_3\beta^3 + O(\beta^4), \tag{2.10}$$

where $\beta = \alpha - \alpha_m$. Moreover, c_m, $\bar{a}_2$ and $\bar{a}_3$ are complex numbers, while $\bar{a}_{1r}$ is a real number and all remain finite as $R \to R_c$. In addition we note that c_{mi}, the imaginary part of c_m, is proportional to $(R - R_c)$ as $R \to R_c$, and that $\bar{a}_{2r} > 0$ when R is close to R_c.

For given α and R the residue at a pole gives the appropriate eigenfunction. And for the situation in which (2.10) is valid, the residue gives an expansion about $\alpha = \alpha_m$ of the eigenfunction of (2.9); if we assume the residue to be regular it may be written

$$F(z;\alpha,R) = \Lambda_0(z;R)+\beta\Lambda_1(z;R)+\beta^2\Lambda_2(z;R)+\ldots, \tag{2.11}$$

where Λ_n is a complex function of z independent of α.

The general form for Ψ_1 is

$$\Psi_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[i\alpha x + s(\alpha)\,t\right] F(z;\alpha,R)\,d\alpha. \tag{2.12}$$

When $t \to \infty$ and α is real the maximum value of the modulus of the integrand of (2.12) occurs at $\alpha = \alpha_m$, and the integrand is then equal to

$$(1/2\pi)\,\Lambda_0(z)\exp\left[i\alpha_m(x-c_{mr}t)+\alpha_m c_{mi}t\right], \tag{2.13}$$

which suggests that Ψ_1 is dominated by an exponentially growing factor when $R > R_c$ and t is large. This deduction may be spurious, however, since what is really significant is the stationary value of $[i\alpha x + s(\alpha)t]$ as a function of α, and this value will normally occur off the real axis. If it occurs at $\overline{\alpha}$ and $s_r(\overline{\alpha}) = p$ then a path of integration may be chosen to pass through $\overline{\alpha}$ and the resulting form of Ψ_1 is dominated by the factor e^{pt} as $t \to \infty$. Clearly $p \leqslant \alpha_m c_{mi}$ so that the exponential factor cannot exceed $e^{\alpha_m c_{mi} t}$. Moreover, $\overline{\alpha}$ is a function of x and t so that strictly Ψ_1 is not normally periodic in x.

In order to obtain some details let us examine the case when $|x+\bar{a}_{1r}t| \ll |a_2 t|$ for then (2.10) may be used to find the value, $\overline{\alpha}$, of α which makes $[i\alpha x + s(\alpha)t]$ stationary. We find

$$\overline{\alpha} = \alpha_m + \frac{i}{2\bar{a}_2 t}(x+\bar{a}_{1r}t) - \frac{3\bar{a}_3}{2\bar{a}_2}\left(\frac{x+\bar{a}_{1r}t}{2\bar{a}_2 t}\right)^2 + O\left(\frac{x+\bar{a}_{1r}t}{\bar{a}_2 t}\right)^3. \tag{2.14}$$

Then the method of steepest descents yields

$$\begin{aligned}\hat{\Psi}_1 \sim \frac{e^{i\bar{\theta}}}{(2\pi)^{\frac{1}{2}}}&\exp\left[i\alpha_m(x-c_m t)\right]\exp\left[-(x+\bar{a}_{1r}t)^2/4\bar{a}_2 t\right]\\ &\times\frac{[\Lambda_0 + i[(x+\bar{a}_{1r}t)/2\bar{a}_2 t]\,\Lambda_1 + O[(x+\bar{a}_{1r}t)/2\bar{a}_2 t]^2]}{|2\bar{a}_2 t|^{\frac{1}{2}}\,|1-(3\bar{a}_3 i/\bar{a}_2)[(x+\bar{a}_{1r}t)/2\bar{a}_2 t] + O[(x+\bar{a}_{1r}t)/2\bar{a}_2 t]^2|^{\frac{1}{2}}},\end{aligned} \tag{2.15}$$

where $\bar{\theta}$ is the direction of steepest descent and terms $O((x+\bar{a}_{1r}t)/2\bar{a}_2 t)^3$ have been omitted from the exponent of the exponential function in (2.15). The leading term of (2.15) has also been obtained by Benjamin (1961) and Gaster (1968*a*, *b*) in another context.

It can be seen that the amplitude of the disturbance reaches a maximum when $|(x+\bar{a}_{1r}t)| \ll |\bar{a}_2 t|$. Indeed the expression (2.15) shows that the 'wave system' moves with the group velocity, which is $(-\bar{a}_{1r})$ in the present problem. If $|(x+\bar{a}_{1r}t)^2/4\bar{a}_2 t|$ becomes large the disturbance diminishes, since $\bar{a}_{2r} > 0$.

An important deduction from (2.15) for the present work is that the disturbance is not periodic in x but has aperiodic properties. Thus some modification of Stuart's (1960) paper is required, since a monochromatic wave was assumed there. Stuart considered a time scale c_i^{-1} for growth of the disturbance, c_i being the imaginary part of a complex wave velocity not necessarily restricted to the case of maximum amplification. For a wave system propagating at the group velocity $(-\bar{a}_{1r})$, the implication of (2.16) is that the length scale also should vary on the scale c_i^{-1}. If this can be done for the non-linear case, it implies a slow variation of amplitude and wavelength of non-linear waves, and negates the assumption of monochromatic waves.

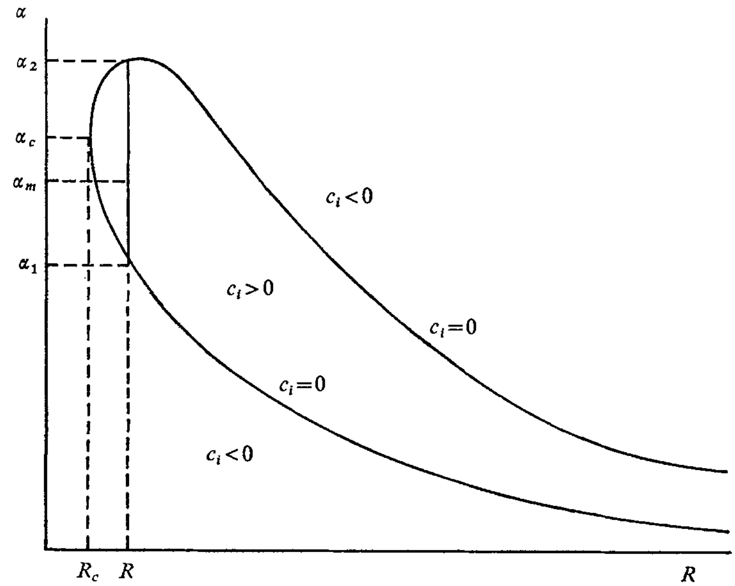

FIGURE 1. Schematic view of neutral wave.

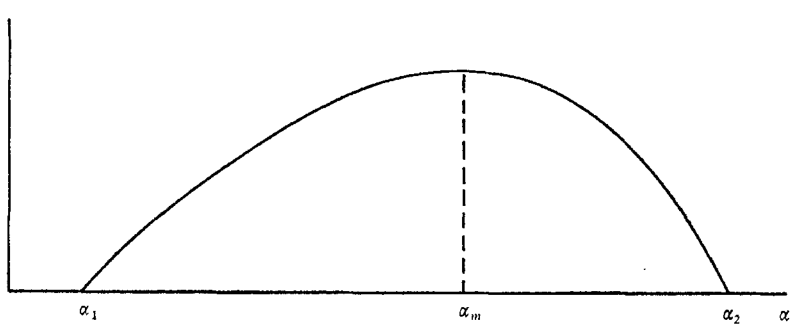

FIGURE 2. Amplification rate for $R > R_c$.

Before going on to discuss the non-linear theory, we shall derive some basic formulae of linearized theory. The sketches in figures 1 and 2 illustrate the properties of αc_i and αc_r that we wish to exploit, particularly near to $R = R_c$. Consider first a perturbation at a Reynolds number R_c and a wave-number α. Then the eigenvalue $s(\alpha)$ is imaginary and equal to $-ic_r\alpha_c$ while the eigenfunction may be defined as $\psi_1(z)$ where

$$\psi_1 = \lim_{R\to R_c} \Lambda_0(z) \tag{2.16}$$

and satisfies

$$(1-z^2-c_r)(\psi_1''-\alpha_c^2\psi_1)+2\psi_1+(i/\alpha_c R_c)(\psi_1^{\text{iv}}-2\alpha_c^2\psi_1''+\alpha_c^4\psi_1) = 0, \tag{2.17}$$

together with the normalizing condition $\psi_1(0) = 1$. We now suppose that $(R-R_c)$ and $(\alpha-\alpha_c)$ are both small and set up a double power series expansion for both $s(\alpha)$ and F, in powers of $(R-R_c)$ and $(\alpha-\alpha_c)$. We write in fact

$$s(\alpha) = -ic_r\alpha_c + ia_{1r}(\alpha-\alpha_c) - a_2(\alpha-\alpha_c)^2 + \ldots + (R-R_c)\,d_1 + \ldots, \qquad (2.18)$$

where terms of the order $(\alpha-\alpha_c)^3$, $(R-R_c)^2$, $(R-R_c)\,(\alpha-\alpha)$ have all been neglected; furthermore a_{1r} is real from the definition of α, while a_2 is complex with a positive real part. Both a_{1r} and a_2 are independent of $(\alpha-\alpha_c)$ and $(R-R_c)$. Simultaneously we write

$$F = \psi_1(z) + (\alpha-\alpha_c)\,\psi_{10}(z) + (R-R_c)\,\psi_{11}(z) + (\dot{\alpha}-\alpha_c)^2\,\psi_{12}(z) + \ldots, \qquad (2.19)$$

where ψ_{10}, ψ_{11}, ψ_{12} are independent of $(\alpha-\alpha_c)$ and $(R-R_c)$. The differential equations satisfied by ψ_{1n} all have the operator of (2.17) on the left-hand side but are non-homogeneous. For the ψ_{10} equation, the right-hand side is

$$-\frac{a_{1r}}{\alpha}(\psi_1''-\alpha_c^2\psi_1) - \frac{2}{\alpha_c}\psi_1 - \left(\frac{1-z^2}{\alpha_c} - \frac{4i}{R_c}\right)(\psi_1''-\alpha^2\psi_1) + 2\alpha_c(1-z^2-c_r)\,\psi_1. \qquad (2.20)$$

Since the homogeneous equation for ψ_{10} is identical with that for ψ_1 of which a solution is known to exist satisfying the no-slip boundary conditions, a solution to the non-homogeneous equation exists only if (2.20) satisfies a certain integral relation. In fact we multiply (2.20) by Φ, which satisfies the differential equation adjoint to (2.17) and the same boundary conditions, as defined by Stuart (1960, p. 374). Then, on integrating with respect to z over the range $-1 < z < 1$, we must get the answer zero and so

$$a_{1r}\int_{-1}^{+1}\Phi(\psi_1''-\alpha_c^2\psi_1)\,dz = \int_{-1}^{+1}dz\,\Phi\left\{2\alpha_c^2(1-z^2-c_r)\,\psi_1 - 2\psi_1 - \left(1-z^2-\frac{4i\alpha_c}{R_c}\right)(\psi_1''-\alpha_c^2\psi_1)\right\}. \qquad (2.21)$$

In the same way, by considering the equation for ψ_{11} we can show that

$$d_1\int_{-1}^{+1}\Phi(\psi_1''-\alpha_c^2\psi_1)\,dz = -\frac{1}{R_c^2}\int_{-1}^{+1}dz\,\Phi\{\psi_1^{\text{iv}} - 2\alpha_c^2\psi_1'' + \alpha_c^4\psi_1\}. \qquad (2.22)$$

The relation for a_2 is more complicated and the computation of ψ_{10} is required before the integrals can be evaluated, as the following formula shows:

$$a_2\int_{-1}^{+1}\Phi(\psi_1''-\alpha_c^2\psi_1)\,dz$$
$$= \int_{-1}^{+1}\Phi\,dz\left\{\begin{array}{l} -i\alpha_c(1-z^2-c_r)\,(2\alpha_c\psi_{10}+\psi_1) + i(1-z^2+a_{1r})\,(\psi_{10}''-\alpha_c^2\psi_{10}-2\alpha_c\psi_1) \\ \qquad + 2i\psi_{10} + \dfrac{2}{R_c}[\psi_1'' - 3\alpha_c^2\psi_1 + 2\alpha_c(\psi_{10}''-\alpha_c^2\psi_{10})]. \end{array}\right\} \qquad (2.23)$$

From calculations kindly made for us by Mr R. R. Cousins of the National Physical Laboratory we have the values

$$\left.\begin{array}{l} a_{1r} = -0{\cdot}384, \\ d_1 = (0{\cdot}17+0{\cdot}80i)\,10^{-5}, \\ a_2 = 0{\cdot}183+0{\cdot}070i; \end{array}\right\} \qquad (2.24)$$

similar values can be inferred also from the papers of Nachtsheim (1964) and Grosch & Salwen (1968).

3. Non-linear theory for wave systems

Our object here is to obtain a form of perturbation solution of (2.1) which is centred round the solution proportional to $\exp i\alpha_c(x-c_r t)$, the mode which is neutral and has maximum 'growth' at $R = R_c$. However, as we found in the last section, an important property of wave systems propagating at the group velocity is that the wave modulates in space and time with x and t having the same slow scale. Now Stuart (1960) used the inverse time scale

$$\epsilon = \alpha_c c_i, \tag{3.1}$$

finding the time-scale for growth in the non-linear problem to be the same as that of the linear case. We want to capitalize on this and also to introduce the formal language of the method of multiple scales. We intend to expand our solution in powers of ϵ, or, what amounts to the same thing, powers of $R-R_c$, since $\epsilon \propto R-R_c$ when $R-R_c$ is small. It is helpful to be precise about c_i and so we define

$$\epsilon = d_{1r}|R-R_c| \quad \text{so that} \quad \frac{\epsilon}{|R-R_c|} = \lim_{\substack{R\to R_c \\ \alpha=\alpha_c}} \frac{\operatorname{Re} s(\alpha)}{R-R_c}. \tag{3.1a}$$

Thus from (2.18) $\epsilon = d_{1r}|R-R_c|$. On the whole we shall be concerned with the unstable situation in which $R > R_c$ but shall occasionally comment on the form taken by the governing equation when $R < R_c$.

Let us introduce the additional variables

$$\chi = \epsilon x, \quad \tau = \epsilon t \tag{3.2}$$

into the problem. This we achieve by the method of multiple scalings. In order to extract the short time scale $((\alpha_c c_r)^{-1})$, we also associate t with x as $(x-c_r t)$ in the following, thus dealing with the propagation of wave components at the wave speed c_r of linearized theory.

It is convenient to follow Stuart (1960) and write

$$\psi = \phi_0(z,\chi,\tau)+\phi_1(z,\chi,\tau)E+\tilde{\phi}_1(z,\chi,\tau)E^{-1}+\phi_2(z,\chi,\tau)E^2 + \tilde{\phi}_2(z,\chi,\tau)E^{-2}+\ldots, \tag{3.3}$$

where $E = \exp[i\alpha_c(x-c_r t)]$. Then it follows that

$$\zeta = -(\phi_0''+\epsilon^2\phi_{0\chi\chi})-(\phi_1''-\alpha^2\phi_1+2i\alpha\epsilon\phi_{1\chi}+\epsilon^2\phi_{1\chi\chi})E-\text{c.c.} - (\phi_2''-4\alpha^2\phi_2+4i\alpha\epsilon\phi_{2\chi}+\epsilon^2\phi_{2\chi\chi})E^2-\text{c.c.}\ldots. \tag{3.4}$$

in which, here and in later evaluations, we have used

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial \chi}, \quad \frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial \tau}. \tag{3.5}$$

Primes denote derivatives with respect to z. Some lengthy algebra yields equations for the harmonic components ϕ_0, ϕ_1, ϕ_2, which are generalizations of (2.5)–(2.7) of Stuart (1960), involving especially derivatives with respect to χ and τ.

Now Stuart's (1960) work indicates that the dominant term in ϕ_1 has amplitude $\epsilon^{\frac{1}{2}}$ in the non-linear problem. Strictly speaking if a representative amplitude of the disturbance at $t = 0$ is $\bar{\epsilon}$, $\hat{\Psi}_1 = O(\bar{\epsilon}t^{-\frac{1}{2}}e^{\epsilon t})$ near the centre of the disturbance when $t \gg 1$, $\tau \ll 1$ from (2.15) ,and so the dominant term in ϕ_1 must be

$$O(\epsilon^{\frac{1}{2}}\bar{\epsilon}\tau^{-\frac{1}{2}}e^{\tau}) \quad \text{near} \quad x+a_{1r}t = 0 \quad \text{when} \quad \tau \ll 1.$$

The factor $\epsilon^{\frac{1}{2}}$ fortuitously matches the order of magnitude $\epsilon^{\frac{1}{2}}$ required by Stuart's theory and the factor $\bar{\epsilon}$ can be absorbed into the origin of t. Thus it is consistent to expect that the solution for $\tau \sim 1$ should match with the solution for $t \sim 1$ in the matching region $t \gg 1$, $\tau \ll 1$.

Following the lead of Stuart (1960) and Watson (1960) we write

$$\left.\begin{aligned} \phi_1 &= \epsilon^{\frac{1}{2}}\phi_{11}(\chi,\tau,z)+\epsilon^{\frac{3}{2}}\phi_{13}(\chi,\tau,z)+O(\epsilon^{\frac{5}{2}}),\\ \phi_2 &= \epsilon\phi_{22}(\chi,\tau,z)+O(\epsilon^2),\\ \phi_0 &= z-\tfrac{1}{3}z^3+\epsilon\phi_{02}(\chi,\tau,z)+O(\epsilon^2), \end{aligned}\right\} \tag{3.6}$$

where ϕ_{nm} are explicitly independent of x, t and ϵ.

In order to allow for the Reynolds number to vary away from R_c, we expand R according to (2.19) but with $\alpha = \alpha_c$; and with reference to (3.1a) this gives

$$R = R_c + d_{ir}^{-1}\epsilon + O(\epsilon^2). \tag{3.6a}$$

On substituting into the fundamental differential equation (2.1), together with (2.2) and equating like powers of ϵ to zero, we obtain, as a first non-trivial result, that

$$(1-z^2-c_r)\left(\frac{\partial^2\phi_{11}}{\partial z^2}-\alpha_c^2\phi_{11}\right)+2\phi_{11}+\frac{i}{\alpha R_c}\left(\frac{\partial^4\phi_{11}}{\partial z^4}-2\alpha_c^2\frac{\partial^2\phi_{11}}{\partial z^2}+\alpha_c^4\phi_{11}\right) = 0 \tag{3.7}$$

from the coefficient of $E\epsilon^{\frac{1}{2}}$; the complex conjugate of (3.7) is obtained from the coefficient of $E^{-1}\epsilon^{\frac{1}{2}}$. The boundary conditions satisfied by ϕ_{11} are that

$$\phi_{11} = \partial\phi_{11}/\partial z = 0 \quad \text{at} \quad z = \pm 1. \tag{3.8}$$

Hence from the linear theory of stability, with $c_r = \lim_{R\to R_c} c_m$ and $\alpha = \alpha_c$, we have

$$\phi_{11} = A(\chi,\tau)\,\psi_1(z), \tag{3.9}$$

where A is an amplitude function to be found.

On equating the coefficient of ϵ to zero we obtain two relations connecting the ϕ_{nm}, one from terms containing E^2 as a factor and one from terms independent of E. These are

$$(1-z^2-c_r)\left(\frac{\partial^2\phi_{22}}{\partial z^2}-4\alpha_c^2\phi_{22}\right)+2\phi_{22}+\frac{i}{2\alpha_c R_c}\left[\frac{\partial^4\phi_{22}}{\partial z^4}-8\alpha_c^2\frac{\partial^2\phi_{22}}{\partial z^2}+16\alpha_c^4\phi_{22}\right]$$
$$= -\tfrac{1}{2}A^2[\psi_1'\psi_1''-\psi_1\psi_1'''] \tag{3.10}$$

and

$$\frac{1}{R_c}\frac{\partial^4\phi_{02}}{\partial z^4} = i\alpha_c A\tilde{A}[\psi_1'\tilde{\psi}_1-\psi_1\tilde{\psi}_1']''. \tag{3.11}$$

Since ϕ_{22} satisfies the same boundary conditions as ϕ_{11}, (3.8), and the results of many numerical calculations strongly suggest that there are no complementary functions of (3.10) which satisfy them, we may take the solution of (3.10) to be unique and write it as

$$\phi_{22} = A^2\psi_2(z). \tag{3.12}$$

Similarly

$$\phi_{02} = |A|^2 F(z). \tag{3.13}$$

In the important case in which $\psi_1(z)$ is even, the solution of (3.11) satisfying conditions of no-slip at $z = \pm 1$ is given by (3.13) with

$$F'(z) = S(z) - \tfrac{3}{2}(1-z^2)\int_0^1 S(z)\,dz; \quad S = i\alpha_c R_c \int_1^\tau (\psi_1' \bar{\psi}_1 - \psi_1 \bar{\psi}_1')\,dz. \tag{3.14}$$

As Watson (1962) shows for $A(\chi,\tau) = A(\chi)$, this implies constancy of mass flux. The case of $A(\chi,\tau) = A(\tau)$ is now seen to be singular since, in contrast to the discussions of Stuart (1960) and Watson (1960) for that case, we no longer have the choice between constancy of mass flux and constancy of pressure gradient. In general, the latter must necessarily change. (We are indebted to Dr M. Gaster for a helpful comment which eliminated an error in an earlier draft.)

Now let us consider the coefficient of $\epsilon^{\frac{3}{2}}$. Equating the terms proportional to $E\epsilon^{\frac{3}{2}}$ to zero we obtain

$$\begin{aligned}(1-z^2-c_r)\left(\frac{\partial^2\phi_{13}}{\partial z^2} - \alpha_c^2\phi_{13}\right) + 2\phi_{13} + \frac{i}{\alpha R_c}\left[\frac{\partial^4\phi_{13}}{\partial z^4} - 2\alpha_c^2\frac{\partial^2\phi_{13}}{\partial z^2} + \alpha_c^4\phi_{13}\right] \\ = \frac{i}{\alpha}\frac{\partial A}{\partial \tau}[\psi_1'' - \alpha_c^2\psi_1] + \frac{iA}{\alpha_c d_{1r} R_c^2}[\chi_1^{\text{iv}} - 2\alpha_c^2\psi_1'' + \alpha_c^4\psi_1] \\ - \frac{i}{\alpha_c}\frac{\partial A}{\partial \chi}\left[2\alpha_c^2(1-z^2-c_r)\psi_1 - \left\{1 - z^2 - \frac{4i\alpha_c}{R_c}\right\}\{\psi_1'' - \alpha_c^2\psi_1\} - 2\psi_1\right] \\ + A|A|^2[\psi_2'(\bar{\psi}_1'' - \alpha_c^2\bar{\psi}_1) + 2\psi_2(\bar{\psi}_1'' - \alpha_c^2\bar{\psi}_1') - 2\bar{\psi}_1'(\psi_2'' - 4\alpha_c^2\psi_2) \\ - \bar{\psi}_1(\psi_2''' - 4\alpha_c^2\psi_2') - F'(\psi_1'' - \alpha_c^2\psi_1) + F'''\psi_1].\end{aligned} \tag{3.15}$$

Since the left-hand side of (3.15) contains the same differential operator as (3.7) and ϕ_{13} satisfies the same boundary condition as ϕ_{11}, it follows that a solution to (3.15) exists only if the right-hand side satisfies a certain integral relation, namely that it be orthogonal to the adjoint function Φ, which satisfies the equation adjoint to (3.7). On multiplying (3.15) by Φ, integrating with respect to z from -1 to $+1$ and using the integral properties of Φ, ψ_1, obtained in §2, we find that for a solution of (3.15) to exist

$$\frac{\partial A}{\partial \tau} - a_{1r}\frac{\partial A}{\partial \chi} = \frac{d_1}{d_{1r}}A + kA|A|^2, \tag{3.16}$$

when $R > R_c$. [If $R < R_c$ it is necessary to change the sign of the term $d_1 A/d_{1r}$ only.] Here a_{1r} is given by (2.21), d_1 by (2.22) and

$$k\int_{-1}^{+1}\Phi(\psi_1'' - \alpha^2\psi_1)\,dz = i\alpha_c\int_{-1}^{+1}\Phi\, g(z)\,dz \tag{3.17}$$

and g is the coefficient of $A|A|^2$ in the right-hand side of (3.15). From the definitions (2.18) and (3.1 a) with $\alpha = \alpha_c$ it follows that

$$\frac{d_1}{d_{1r}} = 1 + i\frac{d_{1i}}{d_{1r}} = 1 - \frac{i}{d_{1r}}\left(\frac{ds_i}{dR}\right)_{\substack{R=R_c\\ \alpha=\alpha_c}}. \tag{3.18}$$

The imaginary part of (3.18) represents the change of wave speed for R different from R_c.

On setting $A = \epsilon^{-\frac{1}{2}}A_1$, $\epsilon = \alpha_c c_i$, $a_{1r} = d_{1i} = 0$, and replacing k by $i\alpha_c k$ we retrieve equation (4.12) of Stuart (1960). For a variety of reasons, however, it is convenient to use (3.2) and to rewrite (3.16) as

$$\frac{\partial A_1}{\partial t} - a_{1r}\frac{\partial A_1}{\partial x} = \epsilon A_1 + a_1 A_1|A_1|^2, \quad R > R_c, \tag{3.19}$$

where

$$A = \epsilon^{-\frac{1}{2}}A_1(t, x)\exp(id_{1i}\epsilon t/d_{1r}). \tag{3.20}$$

[For $R < R_c$, the ϵ terms in (3.19) and in the exponent of (3.20) change sign.] In this form we have explicitly eliminated the part of the coefficient of A_1 which represents the change of wave speed. Thus (3.19) is our first generalization of the amplitude equation derived by Stuart (1960) and adumbrated by Landau (1944).

If, in (3.19) we regard A_1 as a function of t and of

$$X = x + a_{1r}t, \tag{3.21}$$

it takes the form

$$\partial A_1(t, X)/\partial t = \epsilon A_1 + kA_1|A_1|^2, \quad R > R_c, \tag{3.22}$$

which, apart from the 'parameter' X, is the original equation of Stuart (1960). Its solution is clearly given by an adaptation of (5.2) and (5.5) of that paper, namely

$$|A_1|^2 = \frac{\epsilon f^2(X)\,e^{2\epsilon t}}{1 - k_r f^2(X)\,e^{2\epsilon t}}. \tag{3.23}$$

$$A_1 = |A_1|\exp\left\{ik_i\int_{t_0(X)}^{t}|A_1|^2\,dt\right\} \tag{3.24}$$

in which $f^2(X)$ and $t_0(X)$ are arbitrary functions of X. A sign change of the ϵ term deals with $R < R_c$ in (3.22), (3.23).

4. The neighbourhood of $x = -a_{1r}t$

When A_1 is sufficiently small the non-linear terms in (3.19) may be neglected and the equation then has a simple general solution

$$A_1 = \epsilon^{\frac{1}{2}}\,e^{\epsilon t}f(x + a_{1r}t), \tag{4.1}$$

a result which can be inferred from (3.23) and (3.24) in the limit of f^2 small, which yields linearized theory. This solution has a number of features in common with the asymptotic expansion, as $t \to \infty$, of the original small disturbance given in (2.15). For example the role of the group velocity $-a_{1r}$ is brought out by $f(x+a_{1r}t)$ and the exponential growth of the amplitude with time by the factor $e^{\epsilon t}$; moreover the slowly oscillating factor of (3.20) demonstrates the variability with R of the most favoured mode. It is clear however, from (2.15) that a more refined treatment than that leading to (4.1) is necessary if we are to describe the behaviour of A near $x+a_{1r}t=0$. Further examination of (2.15), especially of its exponential factor, suggests that an additional required variable is ξ, defined by

$$\xi = (x+a_{1r}t)\,\epsilon^{\frac{1}{2}} = \epsilon^{\frac{1}{2}}X. \tag{4.2}$$

In terms of τ, ξ the coefficient of $\exp[i\alpha(x-c_r t)]$ in (2.15) reduces to a function proportional to

$$\epsilon^{\frac{1}{2}}\exp\left[\tau-\frac{id_{1i}}{d_{1r}}\tau\right]\;\frac{1}{\tau^{\frac{1}{2}}}\exp\left[-\xi^2/4\alpha_2\tau\right]\psi_1(z)\left[1+O(\epsilon^{\frac{1}{2}})\right] \tag{4.3}$$

when t is large, $\tau,\xi \sim 1$ and $R > R_c$. It is this form which especially suggests the transformation (4.2). Moreover, whereas (3.22) indicates that the variable X of $A_1(t,X)$ may be regarded as a parameter in equation (3.22), it is our object here to allow for *differentials with respect to* X, through the scaled variable ξ.

We now therefore revise (3.3) to have the ϕ_n as functions of z, τ, ξ instead of z, τ, χ and furthermore to allow for the possibility of correction terms of relative order $\epsilon^{\frac{1}{2}}$ to ϕ_{11}, as suggested by (4.3); thus we generalize the expansion formula (3.6) to

$$\phi_1 = \epsilon^{\frac{1}{2}}\phi_{11}(\tau,\xi,z)+\epsilon\phi_{12}(\tau,\xi,z)+\epsilon^{\frac{3}{2}}\phi_{13}(\tau,\xi,z)+O(\epsilon^2). \tag{4.4}$$

The expansion (3.6*a*) for the Reynolds number remains unchanged. At the same time we add in the variable ξ in the expansions of ϕ_2, ϕ_3, and note that the error terms are $O(\epsilon^{\frac{3}{2}})$ instead of $O(\epsilon^2)$. Further (3.5) must be modified to

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial \tau}+b_1\epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi}. \tag{4.5}$$

On substituting into the governing equations, and equating coefficients of $E\epsilon^{\frac{1}{2}}$ to zero we find a result analogous to (3.9), namely

$$\phi_{11} = A(\tau,\xi)\,\psi_1(z) \tag{4.6}$$

where A is now a function of τ, ξ to be found. On equating the coefficients of ϵ to zero, we have, in addition to (3.10) and (3.11) which yield the solutions (3.12), (3.13) as before, yet another equation. This is

$$\begin{aligned}(1-z^2-c_r)\left[\frac{\partial^2\phi_{12}}{\partial z^2}-\alpha_c^2\phi_{12}\right]+2\phi_{12}+\frac{i}{\alpha R_c}\left[\frac{\partial^4\phi_{12}}{\partial z^4}-2\alpha_c^2\frac{\partial^2\phi_{12}}{\partial z^2}+\alpha_c^4\phi_{12}\right]\\ =\frac{i}{\alpha}\frac{\partial A}{\partial \xi}\left[a_{1r}(\psi_1''-\alpha_c^2\psi_1)-2\alpha_c^2\psi_1(1-z^2-c_r)+\left\{1-z^2-\frac{4i\alpha_c}{R_c}\right\}\{\psi_1''-\alpha_c^2\psi_1\}+2\psi_1\right].\end{aligned} \tag{4.7}$$

Since the left-hand side of (4.7) has the same operator as (3.7) a solution is only possible if the right-hand side satisfies a certain integral condition. The presence of a_{1r} on the right-hand side ensures that this *is* possible, whereas, if a similar term to ϕ_{12} had been introduced in (3.6), there would have been no such possibility. Here, by virtue of (2.22) the condition is satisfied for all A; we emphasize that this is not fortuitous, but reflects the natural role of the ξ variable. It follows that ϕ_{12} exists and has the form

$$\phi_{12} = -i(\partial A/\partial\xi)\,\psi_{10}(z)+A_2(\tau,\xi)\,\psi_1(z), \tag{4.8}$$

where A_2 is a function of τ, ξ to be found.

Turning now to the analogue of (3.15) we can see that the terms on the right-hand side proportional to $\partial A/\partial\chi$ will be replaced by two terms; one is proportional to $\partial^2 A/\partial\xi^2$ and has a coefficient which is given by $(-i/\alpha_c\Phi)$ times the integrand on the right-hand side of (2.23); the other is proportional to $\partial A_2/\partial\xi$ with a coefficient identical with that of $\partial A/\partial\xi$ in (4.7). Hence the generalization of (3.16) is

$$\frac{\partial A}{\partial\tau}-a_2\frac{\partial^2 A}{\partial\xi^2}=\frac{d_1}{d_{1r}}A+kA|A|^2, \quad R>R_c, \tag{4.9}$$

where a_2 is the complex number given by (2.23). The reason for the appearance of a_2 is that the method by which the various terms in the expansion of $s(\alpha)$ are evaluated in §2 is equivalent to the method used in this section to obtain the linear version of the differential equation (4.10). For a similar reason the coefficient of $\partial A/\partial\chi$ in (3.16) is $(-a_{1r})$.

Using (3.18) and (3.20) to eliminate the 'wave speed' part of (d_1/d_{1r}) and re-introducing for convenience the t and X variables from (3.2) and (4.2), we then obtain

$$\frac{\partial A_1}{\partial t}-a_2\frac{\partial^2 A_1}{\partial X^2}=\epsilon A_1+kA_1|A_1|^2, \quad R>R_c, \tag{4.10}$$

X being given by (3.21). [For $R<R_c$, the ϵ term changes sign.] This is our second, and more extensive generalization of the amplitude equation of Stuart (1960).

5. Discussion

In the form (4.10) our amplitude equation achieves the object, set out at the beginning of §4, generalizing (3.22) to allow for X differentials. Thus, whereas X in (3.22) is a parameter, in (4.10) it is not. It is worth noting several special features of (4.10) as follows: (*a*) If A_1 is a function of t only, (4.10) reduces to the amplitude equation of Stuart (1960). (*b*) If X is kept constant, (4.10) reduces to our first generalization (3.19) but in the form (3.22). (*c*) As a consequence of X being constant, an equivalent form of (3.22) is

$$-a_{1r}\frac{\partial A_1(x,X)}{\partial x}=\epsilon A_1+kA_1|A_1|^2 \tag{5.1}$$

which is Watson's (1962) equation for spatially growing waves. (*d*) Equations (3.22) and (5.1) in their linear forms together illustrate Gaster's (1962) theorem that time-growing waves can be converted into spatially growing waves, in particular by division of the expression for $\epsilon=\alpha c_i$ by the group velocity $(-a_{1r})$. (*e*) There is a formal similarity between our equation (4.10) and the equation for thermal convection derived somewhat arbitrarily by Newell & Whitehead (1969) and with greater justification by Segel (1969) simultaneously and independently. In that problem, however, the group velocity is zero, so that X there is simply a spatial variable. We regard the derivation of (4.10) for cases in which the group velocity is not zero, that is for propagating waves, as the most interesting result of the present paper. The crucial factor is the dominant role of the variable $X=(x+a_{1r}t)$ which, if it truly varies, plays the part assigned to it by (4.10); on the other hand if X is constant the relevant equation is (3.22) which represents a form of our first generalization (3.19). Of especial importance is the fact that the linear form of (4.10) allows solutions of the form (2.15) which develop from initially prescribed distributions. Because of this it is thought that the generalization of the amplitude equation which (4.10) affords is of significance.

In view of the complexity of the arguments used in the present paper, and especially because of the results (3.22) and (4.10) derived here, we will comment further on the case in which both the wave velocity and the group velocity are zero. Examples are Bénard convection mentioned above and the instability which gives rise to Taylor vortices between rotating cylinders. Clearly our arguments, based on (2.15), which suggest that for disturbances propagating at the group velocity the length and time scales should be equal for 'slow' amplitude modulations, do not apply at all if $a_{1r} = 0$. For the Bénard and Taylor problems mentioned above, different length and time scales can be expected to be appropriate. For example, in the Bénard problem, Newell & Whitehead (1969) and Segel (1969) use a length scale $\epsilon^{-\frac{1}{2}}$ and a time scale ϵ^{-1}, the latter being the growth rate; although Newell & Whitehead assumed this, Segel gives arguments indicating that the term $\partial A_1/\partial x$ disappears from the amplitude equation. However, the reason is *not* the same as the one by which our equation (3.19) can be converted to the form (3.22). Both papers obtain a term proportional to $\partial^2 A_1/\partial x^2$.

More recently Di Prima, Eckhaus & Segel (1971) have made a thorough analysis of the non-linear interaction of a denumerable set of discrete modes spread over the neighbourhood of $\alpha = \alpha_c$. They find that if $a_{1r} = 0$ "the most dangerous band of wave-numbers is when $\alpha - \alpha_c = O(\epsilon^{\frac{1}{2}})$, which should be covered with modes spaced $O(\epsilon^{\frac{1}{2}})$ apart" and that their properties are essentially controlled by the same partial differential equation as that of Newell & Whitehead (1969) and Segel (1969). If $a_{1r} \neq 0$ "the band of wave-number when $\alpha - \alpha_c = O(\epsilon^{\frac{1}{2}})$ should be covered with modes spaced $O(\epsilon)$ apart". The basic differential equation is the same as (3.16) in the present paper and, if X differentials are allowed, (4.10) can be derived. [The quotations given above are due to Professor Di Prima.]

Although the derivation of (4.10) assumes $R > R_c$, this feature is not essential. If $R < R_c$ the maximum 'growth' rate of figure 2 is negative, and the signs of A_1 on the right-hand sides of (3.22), (4.10) need changing. Moreover, the mathematics given in this paper is not especially restricted to plane Poiseuille flow, since the replacement of $(1-z^2)$ by $\overline{u}$ is easily achieved. Other flows, parallel or approximately parallel in some sense, may be discussed by the methods described here. If the problem for some $\overline{u}$ yields $k_r < 0$ in the neighbourhood of $\alpha = \alpha_c$, then (3.23) shows that for X constant and for $R > R_c\,|A_1|^2$ tends to the equilibrium value $(-\epsilon/k_r)$ when $t \to \infty$. But if $k_r < 0$ and $R < R_c$, (3.22) shows that the rate of damping is increased by the non-linear term.

Plane Poiseuille flow is different from this, since the work of Pekeris & Shkoller (1967) and Reynolds & Potter (1967) shows that, for a wide range of Reynolds numbers centred around R_c and for a range of wave-numbers centred around α_c, the parameter $k_r > 0$. Thus, if $R > R_c$ the rate of growth is increased by the presence of the non-linear term. Of more important consequence, however, is the fact that, if $R < R_c$, an equilibrium solution of (4.10) is $|A_1|^2 = \epsilon/k_r$, at least when X is constant. This result gives a threshold amplitude for $R < R_c$ and α

in some neighbourhood of α_c, such that instability occurs for amplitudes $|A_1|^2$ greater than (ϵ/k_r). The amplitude equation (4.10) eventually ceases to be valid, since the value of $|A_1|^2$ increases without bound.

The above result is, of course, well known by now in the case when $a_{1r} = 0$ in (3.16) and our comments above merely extend it to the case of X constant. The present work is an extension of the original time-dependent amplitude equation of Stuart (1960) and not only because it allows the wave system to be spatially modulated. There is a deeper reason. This earlier theory is virtually the only theory of non-linear instability which can make a serious claim to be rational in Van Dyke's (1964) sense and therefore mathematically consistent. However, it suffers from the defect that the initial disturbance must be spatially unmodulated so that on $\tau \to 0$, A is independent of X. The initial infinitesimal disturbance that we consider in § 2 is quite general and, as $t \to \infty$ (with $\epsilon t \ll 1$), leads to a spatially modulated wave (2.15). This must be and can be matched to the solution of (4.9) as $\tau \to 0$. The necessary initial condition on A is that

$$A \approx \frac{\Delta}{\tau^{\frac{1}{2}}} e^{-\xi^2/4a_2\tau} \quad \text{as} \quad \tau \to 0, \tag{5.2}$$

where Δ ($\ll 1$) is a constant of the order of the original infinitesimal disturbance to the plane parallel flow. On adding the boundary condition $|A| \to 0$ as $|\xi| \to \infty$ the determination of A satisfying (4.9) presents a well-posed problem. Its study is at present under way and we hope to report on the results in the near future.

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